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MEAN CONVERGENCE OF GENERALIZED WALSH-FOURIER SERIES

BY

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ABSTRACT. Paley proved that Walsh-Fourier series converges in L^p $(1 < p < \infty)$. We generalize Paley's result to Fourier series with respect to characters of countable direct products of finite cyclic groups of arbitrary orders.

1. Introduction. It is known that the Walsh functions are characters of the countable direct product of groups of order 2. In this note we consider characters of $\Pi_{i=0}^{\infty} Z_{p_i}$, where Z_{p_i} is a cyclic group of order p_i, $p_i \geqslant 2$. Various Fourier properties of this generalized Walsh system have been studied in [8], [7], [9], [5], [3], [4], [2], and others. Many of these results are obtained only for the case where $\sup_i p_i < \infty$. In fact, Price [7] showed that some basic properties no longer hold when $\sup_i p_i = \infty$. We will show that results concerning mean convergence, however, are still valid even if the orders p_i are unbounded. The bounded case was first obtained by Watari [9]. See also Gosselin [2].

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Let $\{p_i\}_{i \geqslant 0}$ be a sequence of integers, $p_i \geqslant 2$. Let $G = \Pi_{i=0}^{\infty} Z_{p_i}$ be the direct product of cyclic groups of order p_i, and μ the Haar measure on G normalized by $\mu(G) = 1$. Each element of G can be considered as a sequence $\{x_i\}$, with $0 \leqslant x_i < p_i$. Set $m_0 = 1$, $m_k = \Pi_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. We can identify G with the unit interval (0, 1). This identification consists in associating with each $\{x_i\} \in G$, $0 \leqslant x_i < p_i$, the point $\Sigma_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure preserving.

We define an orthonormal system of functions $\{\phi_k\}$ on G. For each $x = \{x_i\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k / p_k)$, $k = 0, 1, \ldots$. We enumerate the set of all finite products of $\{\phi_k\}$ using a scheme of Paley. We express each nonnegative integer n as a finite sum $n = \Sigma_{k=0}^{\infty} \alpha_k m_k$, with $0 \leqslant \alpha_k < p_k$, and define $\chi_n = \Pi_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G. For the case $p_i = 2$, $i = 0, 1, \ldots$, G is the dyadic group, $\{\phi_k\}$ are the Rademacher functions, and $\{\chi_n\}$ the Walsh functions.

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We consider Fourier series with respect to $\{\chi_n\}$. Let $D_n = \Sigma_{j=0}^{n-1}\chi_j$, $n = 1, 2, \ldots$, be the nth Dirichlet kernel. For $f \in L^1(G)$,

$$S_n f(x) = \int_G f(t) D_n(x - t)\, d\mu(t), \qquad n = 1, 2, \ldots,$$

denotes the nth partial sum of the Fourier series of f. We have the following uniform estimates on $\{S_n f\}$.

THEOREM 1. *There are absolute constants C and C_p such that, for $n = 1, 2, \ldots$,*

(1) $$\|S_n f\|_p \leq C_p \|f\|_p, \qquad f \in L^p(G),\ 1 < p < \infty,$$

(2) $$\mu\{|S_n f| > y\} \leq C y^{-1} \|f\|_1, \qquad f \in L^1(G),\ y > 0.$$

These results and the density of the generalized Walsh polynomials imply the mean convergence of $S_n f$ to f in $L^p(G)$, $1 < p < \infty$.

The constants C and C_p in the above theorem are independent of the orders p_i of the cyclic groups.

If $p_i = 2$, $i = 0, 1, \ldots$, Theorem 1 is Paley's result for the Walsh-Fourier series [6]. On the other hand, if $p_0 \longrightarrow \infty$, $S_n f$ resembles the nth trigonometric partial sum. Thus, when restricted to one cyclic group, Theorem 1 can be viewed as a discrete analogue of M. Riesz's theorem for the trigonometric Fourier series [10, I, p. 266].

In what follows C will denote an absolute constant, which may vary from line to line.

2. Modified partial sums and conjugate functions. We will use the following notation. Let $\{G_k\}$ be a sequence of subgroups of G defined by

$$G_0 = G, \quad G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \qquad k = 1, 2, \ldots.$$

Then $\mu(G_k) = m_k^{-1}$. Let $\mathcal{F}_k$ be the σ-algebra generated by the cosets of G_k. On the interval $(0, 1)$, atoms of $\mathcal{F}_k$ are intervals of the form $(jm_k^{-1}, (j+1)m_k^{-1})$, $j = 0, 1, \ldots, m_k - 1$. We note that ϕ_k is measurable with respect to $\mathcal{F}_{k+1}$.

It is proved in [8] that

(3) $$D_{m_k}(x) = \begin{cases} m_k & \text{if } x \in G_k, \\ 0 & \text{otherwise.} \end{cases}$$

From (3) it follows that

$$S_{m_k} f(x) = \frac{1}{\mu(I)} \int_I f\, d\mu,$$

where $I = x + G_k$.

It is also proved in [8] that if $n = \Sigma_{k=0}^{\infty} \alpha_k m_k$, $0 \leqslant \alpha_k < p_k$,

$$D_n = \chi_n \sum_{k=0}^{\infty} D_{m_k} \phi_k^{-\alpha_k} \left(\sum_{j=0}^{\alpha_k - 1} \phi_k^j \right), \tag{4}$$

with the interpretation that $\Sigma_{j=0}^{\alpha_k - 1} \phi_k^j = 0$ if $\alpha_k = 0$. It is convenient to consider the modified Dirichlet kernel D_n^* defined by $D_n^* = \overline{\chi}_n D_n$. From (4) we have

$$D^*_{\alpha_k m_k} = D_{m_k} \phi_k^{-\alpha_k} \left(\sum_{j=0}^{\alpha_k - 1} \phi_k^j \right) = D_{m_{k+1}} - D_{(p_k - \alpha_k) m_k}, \tag{5}$$

and

$$D_n^* = \sum_{k=0}^{\infty} D^*_{\alpha_k m_k}. \tag{6}$$

Let $S_n^* f(x) = \int_G f(t) D_n^*(x - t)\, d\mu(t)$ be the nth modified partial sum. Since $S_n^* f = \overline{\chi}_n S_n(f \chi_n)$, Theorem 1 is equivalent to

THEOREM 1*. *There are absolute constants C and C_p such that, for $n = 1, 2, \ldots,$*

$$\|S_n^* f\|_p \leqslant C_p \|f\|_p, \qquad f \in L^p(G),\ 1 < p < \infty, \tag{7}$$

$$\mu\{|S_n^* f| > y\} \leqslant C y^{-1} \|f\|_1, \qquad f \in L^1(G),\ y > 0. \tag{8}$$

We will prove Theorem 1*. The following facts concerning the modified partial sums will be needed. First of all we have, by (5) and (6),

$$S_n^* f = \sum_{k=0}^{\infty} S^*_{\alpha_k m_k} f, \tag{9}$$

with $S^*_{\alpha_k m_k} f = S_{m_{k+1}} f - S_{(p_k - \alpha_k) m_k} f$. Moreover, it follows from (5) and (3) that

$$S^*_{\alpha_k m_k} f(x) = \frac{1}{\mu(I)} \int_I f(t) \phi_k^{-\alpha_k}(x - t) \left(\sum_{j=0}^{\alpha_k - 1} \phi_k^j(x - t) \right) d\mu(t), \tag{10}$$

where $I = x + G_k$. Now, for $f \in L^1(G)$,

$$\frac{1}{\mu(I)} \int_I f(t) \left(\sum_{j=0}^{\alpha_k - 1} \phi_k^j(x - t) \right) d\mu(t)$$

resembles the α_kth partial sum of the trigonometric Fourier series of f on the coset I. The relation between the trigonometric partial sum and conjugate function leads to our definition of the conjugate function $H_k f$ of $f \in L^1(G)$. Let $x = \{x_k\} \in G$. We define

$$H_k f(x) = \frac{1}{2} \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq t_k\}} f(t) \cot(\pi(x_k - t_k)/p_k)\, d\mu(t),$$

where $I = x + G_k$. Since

$$\phi_k^{-\alpha_k}(t) \sum_{j=0}^{\alpha_k - 1} \phi_k^j(t) = \begin{cases} \alpha_k & \text{if } t_k = 0, \\ \tfrac{1}{2}\phi_k^{-\alpha_k}(t) - \tfrac{1}{2} + \tfrac{1}{2} i \phi_k^{-\alpha_k}(t) \cot(\pi t_k/p_k) & \\ \qquad - \tfrac{1}{2} i \cot(\pi t_k/p_k) & \text{if } t_k \neq 0, \end{cases}$$

(10) implies

$$\begin{aligned} S^*_{\alpha_k m_k} f(x) = {} & \frac{\alpha_k}{\mu(I)} \int_{I \cap \{x_k = t_k\}} f(t)\, d\mu(t) \\ & + \frac{1}{2} \phi_k^{-\alpha_k}(x) \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq t_k\}} f(t) \phi_k^{\alpha_k}(t)\, d\mu(t) \\ & - \frac{1}{2} \frac{1}{\mu(I)} \int_{I \cap \{x_k \neq t_k\}} f(t)\, d\mu(t) \\ & + i \phi_k^{-\alpha_k}(x) H_k(f \phi_k^{\alpha_k})(x) - i H_k f(x). \end{aligned} \tag{11}$$

(9) and (11) will be used later in the proof of Theorem 1*.

3. A decomposition lemma. For the proof of Theorem 1* we need a modified form of the Calderón-Zygmund decomposition lemma [1, p. 91]. The following may best be described on the interval (0, 1).

LEMMA 2. *Let f belong to $L^1(G)$ and $y > 0$ with $\|f\|_1 \leq y$. Let $\{\alpha_k\}_{k \geq 0}$ be a sequence of integers with $0 \leq \alpha_k < p_k$. Then there are L^1 functions g and b, and a collection $\mathcal{C} = \{\omega_j\}$ of disjoint intervals such that*

(12) $f = g + b$.

(13) $|g| \leq Cy$ *a.e.*

(14) $\|g\|_1 \leq C\|f\|_1$.

(15) $\mathcal{C} = \bigcup_{k=0}^{\infty} \mathcal{C}_k$ *where each $\omega_j \in \mathcal{C}_k$ is measurable with respect to $\mathcal{F}_{k+1}$ and is a proper subset of a coset of G_k.*

(16) $b(x) = 0$ *if* $x \notin \bigcup_j \omega_j$.

(17) $\int_{\omega_j} b\, d\mu = 0$ *for every* $\omega_j \in \mathcal{C}$ *and* $\int_{\omega_j} b \phi_k^{\alpha_k} d\mu = 0$ *for every* $\omega_j \in \mathcal{C}_k$, $k = 0, 1, \ldots$.

(18) $\int_{\omega_j} |b|\, d\mu \leq C \int_{\omega_j} |f|\, d\mu$ *for every* $\omega_j \in \mathcal{C}$.

(19) $\Sigma_j \mu(\omega_j) \leq y^{-1} \|f\|_1$.

PROOF. We first construct the collection $\mathcal{C}$ of disjoint intervals. We divide (0, 1) into two subintervals I_1 and I'_1, with $I_1, I'_1 \in \mathcal{F}_1$ and $\mu(I_1) - m_1^{-1} \leq \mu(I'_1) \leq \mu(I_1)$. If $(1/\mu(I_1)) \int_{I_1} |f|\, d\mu > y$, then I_1 is in $\mathcal{C}$. Otherwise we repeat the above process with (0, 1) replaced by I_1. We do the same with I'_1. Finally we reach a stage where the subinterval I is an atom of $\mathcal{F}_1$ and $(1/\mu(I)) \int_I |f|\, d\mu \leq y$.

We then divide I into subintervals I_2 and I_2', with $I_2, I_2' \in F_2$ and $\mu(I_2) - m_2^{-1} \leq \mu(I_2') \leq \mu(I_2)$, and proceed as before. In this way we obtain a collection $C = \{\omega_j\}$ of disjoint intervals which has the properties that

(20) $$y < \frac{1}{\mu(\omega_j)} \int_{\omega_j} |f|\, d\mu \leq 3y, \qquad \omega_j \in C,$$

and

(21) $$|f(x)| \leq y \quad \text{for a.e. } x \notin \bigcup_j \omega_j.$$

The first inequality of (20) implies (19). Set

$$C_0 = \{\omega_j \in C : \omega_j \in F_1\},$$

and

$$C_k = \left\{\omega_j \in C \setminus \bigcup_{l=0}^{k-1} C_l : \omega_j \in F_{k+1}\right\},$$

$k = 1, 2, \ldots$. Then $\{C_k\}$ satisfies (15).

Next we decompose f as $f = g + b$, with

(22) $$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_j \omega_j, \\ a_{kj} + b_{kj}\phi_k^{-\alpha_k}(x) & \text{if } x \in \omega_j \in C_k, \end{cases}$$

where a_{kj}, b_{kj} are constants chosen in such a way that

(23) $$\int_{\omega_j} f\, d\mu = \int_{\omega_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k})\, d\mu,$$

and

(24) $$\int_{\omega_j} f\phi_k^{\alpha_k}\, d\mu = \int_{\omega_j} (a_{kj} + b_{kj}\phi_k^{-\alpha_k})\phi_k^{\alpha_k}\, d\mu.$$

Then $b = g - f$ automatically satisfies (16) and (17). The proof will be completed if we show

(25) $$|g(x)| \leq \frac{C}{\mu(\omega_j)} \int_{\omega_j} |f|\, d\mu, \qquad x \in \omega_j,\ \omega_j \in C,$$

for then (25) together with (20) and (21) will imply (13), (14) and (18).

To prove (25) we write $\beta_k = \alpha_k$ if $0 \leq \alpha_k \leq p_k/2$ and $\beta_k = \alpha_k - p_k$ if $p_k/2 < \alpha_k < p_k$. Then $-p_k/2 < \beta_k \leq p_k/2$ and $\phi_k^{\alpha_k} = \phi_k^{\beta_k}$. Let $\omega_j \in C_k$. If ω_j is a coset of G_{k+1}, or if $\beta_k = 0$, then ϕ_k is constant in ω_j. In this case we set $a_{kj} = (\mu(\omega_j))^{-1}\int_{\omega_j} f\, d\mu$ and $b_{kj} = 0$. (25) follows immediately.

Now suppose $\beta_k \neq 0$ and ω_j is not a coset of G_{k+1}, that is $\mu(\omega_j)m_{k+1} \geq 2$. Then $|(\mu(\omega_j))^{-1}\int_{\omega_j}\phi_k^{\beta_k}\, d\mu| \neq 1$. Solving (23), (24) for a_{kj}, b_{kj} and substituting into (22) we obtain, for $x \in \omega_j$,

$$g(x) = \left[\frac{1}{\mu(\omega_j)}\int_{\omega_j} f\,d\mu - \frac{1}{\mu(\omega_j)}\int_{\omega_j}\phi_k^{-\beta_k}\,d\mu \frac{1}{\mu(\omega_j)}\int_{\omega_j} f\phi_k^{\beta_k}\,d\mu \right.$$
$$+ \frac{1}{\mu(\omega_j)}\int_{\omega_j} f\phi_k^{\beta_k}\,d\mu\phi_k^{-\beta_k}(x)$$
$$\left. - \frac{1}{\mu(\omega_j)}\int_{\omega_j}\phi_k^{\beta_k}\,d\mu\,\frac{1}{\mu(\omega_j)}\int_{\omega_j} f d\mu\,\phi_k^{-\beta_k}(x)\right]$$
$$\times\left[1 - \left|\frac{1}{\mu(\omega_j)}\int_{\omega_j}\phi_k^{\beta_k}\,d\mu\right|^2\right]^{-1}$$
$$= \left[\frac{1}{\mu(\omega_j)}\int_{\omega_j} f(y)\,\frac{1}{\mu(\omega_j)}\int_{\omega_j}(\phi_k^{\beta_k}(y) - \phi_k^{\beta_k}(t))\right.$$
$$\left.\times(\phi_k^{-\beta_k}(x) - \phi_k^{-\beta_k}(t))\,d\mu(t)\,d\mu(y)\right]$$
$$\times\left[1 - \left|\frac{1}{\mu(\omega_j)}\int_{\omega_j}\phi_k^{\beta_k}\,d\mu\right|^2\right]^{-1}.$$

Observe that for $s, t \in \omega_j$,

$$|\phi_k^{\beta_k}(s) - \phi_k^{\beta_k}(t)| \leq |2\pi\beta_k/p_k|\;|s_k - t_k|$$
$$\leq (2\pi|\beta_k|/p_k)\mu(\omega_j)m_{k+1} = 2\pi|\beta_k|\mu(\omega_j)m_k,$$

and

$$|\phi_k^{\beta_k}(s) - \phi_k^{\beta_k}(t)| \leq 2.$$

Also,

$$\left|\frac{1}{\mu(\omega_j)}\int_{\omega_j}\phi_k^{\beta_k}\,d\mu\right| = \left|\frac{1 - \exp(2\pi i\beta_k\mu(\omega_j)m_k)}{\mu(\omega_j)m_{k+1}(1 - \exp(2\pi i\beta_k/p_k))}\right|.$$

Therefore, for $x \in \omega_j$,

$$|g(x)| \leq \frac{1}{\mu(\omega_j)}\int_{\omega_j}|f|\,d\mu\,\min(4, (2\pi\beta_k\mu(\omega_j)m_k)^2)$$
$$\times\left[1 - \left|\frac{1 - \exp(2\pi i\beta_k\mu(\omega_j)m_k)}{\mu(\omega_j)m_{k+1}(1 - \exp(2\pi i\beta_k/p_k))}\right|^2\right]^{-1}. \tag{26}$$

A direct calculation shows that for any integer $n \geq 2$ and any number θ with $-\pi < \theta \leq \pi$, we have

$$(n\theta)^2\,[1 - |(1 - e^{in\theta})/n(1 - e^{i\theta})|^2]^{-1} \leq C \tag{27}$$

for $n|\theta| \leq \pi/10$, and

$$(28) \qquad [1 - |(1 - e^{in\theta})/n(1 - e^{i\theta})|^2]^{-1} \leqslant C$$

for $n|\theta| \geqslant \pi/10$. (25) now follows immediately from (26), (27) and (28). This concludes the proof of the lemma.

4. **Proof of Theorem 1*.** The case $p = 2$ of (7) is a consequence of Plancherel's formula. It therefore suffices to prove (8), for then (7) will follow by the Marcinkiewicz interpolation theorem [10, II, p. 112] and a duality argument.

For the proof of (8) we note that there is nothing to prove if $\|f\|_1 > y$, so we can assume $\|f\|_1 \leqslant y$. Decompose f as in Lemma 2. Since

$$\mu\{|S_n^* f| > y\} \leqslant \mu\{|S_n^* g| > y/2\} + \mu\{|S_n^* b| > y/2\},$$

(8) will follow if we can show that each term on the right is bounded by $Cy^{-1}\|f\|_1$.

Using the fact that $\{S_n^*\}$ is uniformly bounded in L^2, we obtain

$$\mu\{|S_n^* g| > y/2\} \leqslant Cy^{-2}\|S_n^* g\|_2^2 \leqslant Cy^{-2}\|g\|_2^2 \leqslant Cy^{-1}\|f\|_1,$$

by (13) and (14).

To estimate $|S_n^* b|$ we use the following notation. Let $\omega_j \in F_{k+1}$, with ω_j contained in the coset I of G_k. We consider I as a circle, and let ω_j^* denote the interval inside I which contains ω_j at its center and $\mu(\omega_j^*) = 3\mu(\omega_j)$. Let $\Omega^* = \bigcup_j \omega_j^*$. We have, by (19),

$$\mu(\Omega^*) \leqslant 3 \sum_j \mu(\omega_j) \leqslant 3y^{-1}\|f\|_1.$$

Therefore it suffices to prove

$$(29) \qquad \mu\{x \notin \Omega^*\colon |S_n^* b| > y/2\} \leqslant Cy^{-1}\|f\|_1.$$

To do this we expand $S_n^* b$ as in (9) and (11). Moreover, we observe that for $x \notin \Omega^*$ the first three terms in (11) vanish. This can be seen as follows. Let $I = x + G_k$ and $I' = x + G_{k+1}$. Then neither I nor I' is contained in $\bigcup_j \omega_j$. For the first term in (11), we have

$$\int_{I \cap \{x_k = t_k\}} b(t)\, d\mu(t) = \sum_{\omega_j \subset I'} \int_{\omega_j} b\, d\mu = 0,$$

by (16) and (17). For the second term,

$$\begin{aligned}\int_{I \cap \{x_k = t_k\}} b(t)\phi_k^{\alpha_k}(t)\, d\mu(t) &= \sum_{\omega_j \subset I;\, \omega_j \not\subset I'} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\, d\mu(t) \\ &= \sum_{\omega_j \subset I;\, \omega_j \in C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\, d\mu(t) \\ &\quad + \sum_{\omega_j \subset I;\, \omega_j \not\subset I';\, \omega_j \notin C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t) d\mu(t).\end{aligned}$$

If $\omega_j \in C_k$, then $\int_{\omega_j} b\phi_k^{\alpha_k}\, d\mu = 0$, by (17). If $\omega_j \subset I$ and $\omega_j \notin C_k$, then $\phi_k^{\alpha_k}$ is constant on ω_j, so $\int_{\omega_j} b\phi_k^{\alpha_k}\, d\mu = 0$ by (17). Hence $\int_{I\cap\{x_k\neq t_k\}} b(t)\phi_k^{\alpha_k}(t)\, d\mu(t) = 0$. Similarly $\int_{I\cap\{x_k\neq t_k\}} b(t)\, d\mu(t) = 0$. Therefore we have

$$S^*_{\alpha_k m_k} b(x) = i\phi_k^{-\alpha_k}(x) H_k(b\phi_k^{\alpha_k})(x) - iH_k b(x), \qquad x \notin \Omega^*. \tag{30}$$

Thus, if $x \notin \Omega^*$,

$$|S_n^* b(x)| \leq \sum_{k=0}^{\infty} |S^*_{\alpha_k m_k} b(x)| \leq \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| + \sum_{k=0}^{\infty} |H_k b(x)|.$$

(29) will be proved if we can show

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| > \frac{y}{4}\right\} \leq Cy^{-1}\|f\|_1 \tag{31}$$

and

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty} |H_k b(x)| > \frac{y}{4}\right\} \leq Cy^{-1}\|f\|_1. \tag{32}$$

We will demonstrate (31). (32) can be proved similarly.

Suppose $x \notin \Omega^*$. Let $I = x + G_k$ and $I' = x + G_{k+1}$. Then, as before, we have

$$\begin{aligned} H_k(b\phi_k^{\alpha_k})(x) &= \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\, \omega_j \not\subset I'} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t) \\ &= \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\, \omega_j\in C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t) \\ &\quad + \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\, \omega_j\not\subset I';\, \omega_j\notin C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t)\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) d\mu(t). \end{aligned}$$

Again, if $\omega_j \subset I$ and $\omega_j \notin C_k$, $\phi_k^{\alpha_k}(t)\cot(\pi(x_k - t_k)/p_k)$ is constant on ω_j. Therefore the last term on the right vanishes by (17). Moreover, if $\omega_j \in C_k$, $\int_{\omega_j} b\phi_k^{\alpha_k}\, d\mu = 0$, also by (17). Consequently,

$$\begin{aligned} H_k(b\phi_k^{\alpha_k})(x) &= \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\, \omega_j\in C_k} \int_{\omega_j} b(t)\phi_k^{\alpha_k}(t) \\ &\qquad \times \left[\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right] d\mu(t), \end{aligned}$$

where $t^j = \{t_k^j\}_{k\geq 0}$ is any fixed point in ω_j. Thus for any coset I of G_k,

$$\int_{I\cap {}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})(x)|\, d\mu(x)$$

$$\leqslant \frac{1}{2}\frac{1}{\mu(I)} \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |b(t)| \int_{I\cap {}^c\Omega_j^*} \left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x)\, d\mu(t).$$

A simple calculation shows that, for $t \in \omega_j$,

$$\frac{1}{\mu(I)} \int_{I\cap {}^c\omega_j^*} \left|\cot\left(\frac{\pi(x_k - t_k)}{p_k}\right) - \cot\left(\frac{\pi(x_k - t_k^j)}{p_k}\right)\right| d\mu(x) \leqslant C,$$

so we obtain

$$\int_{I\cap {}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})|\, d\mu \leqslant C \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |b|\, d\mu \leqslant C \sum_{\omega_j\subset I;\omega_j\in C_k} \int_{\omega_j} |f|\, d\mu,$$

by (18). Therefore

$$\mu\left\{x \notin \Omega^*: \sum_{k=0}^{\infty} |H_k(b\phi_k^{\alpha_k})(x)| > y/4\right\}$$

$$\leqslant Cy^{-1} \sum_{k=0}^{\infty} \int_{{}^c\Omega^*} |H_k(b\phi_k^{\alpha_k})|\, d\mu \leqslant Cy^{-1} \sum_{k=0}^{\infty} \sum_{\omega_j\in C_k} \int_{\omega_j} |f|\, d\mu$$

$$= Cy^{-1} \sum_j \int_{\omega_j} |f|\, d\mu \leqslant Cy^{-1} \|f\|_1.$$

This establishes (31), and hence completes the proof of Theorem 1*.

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